Math 155, *Lecture Notes- Bonds* **Name**____________

Section 8.3 Trigonometric Integrals

I hope $\sin^2(\theta)+\cos^2(\theta)=1$ is your favorite trigonometric identity!
Frequently, we'll be using this identity and "power reducing" identities,

$$\sin^2(\theta)=\frac{1-\cos(2\theta)}{2} \ \& \ \cos^2(\theta)=\frac{1+\cos(2\theta)}{2},$$

in combination with the substitution technique to evaluate trigonometric integrals of the following form:

Type I: $\int \sin^m(x)\cos^n(x)dx$.

Guidelines for Evaluating Integrals Involving Sine and Cosine

1. If the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosines. Then, expand and integrate.

$$\int \overbrace{\sin^{2k+1}}^{\text{Odd}} x \cos^n x\,dx = \int \overbrace{(\sin^2 x)^k}^{\text{Convert to cosines}} \cos^n x \overbrace{\sin x\,dx}^{\text{Save for } du} = \int (1-\cos^2 x)^k \cos^n x \sin x\,dx$$

2. If the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sines. Then, expand and integrate.

$$\int \sin^m x \overbrace{\cos^{2k+1}}^{\text{Odd}} x\,dx = \int \sin^m x \overbrace{(\cos^2 x)^k}^{\text{Convert to sines}} \overbrace{\cos x\,dx}^{\text{Save for } du} = \int \sin^m x(1-\sin^2 x)^k \cos x\,dx$$

3. If the powers of both the sine and cosine are even and nonnegative, make repeated use of the identities

$$\sin^2 x = \frac{1-\cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

to convert the integrand to odd powers of the cosine. Then proceed as in guideline 2.

n is odd, use (2)

Ex.1 Integrate: $\int \cos^3(x)\sin^4(x)dx = \int \cos^2(x)\sin^4(x)\cdot\cos(x)dx$

"odd guy out"

$= \int [1-\sin^2(x)]\cdot\sin^4(x)\cdot\cos(x)dx$

$= \int (1-u^2)(u^4)\cdot[\cos(x)]\cdot\left(\frac{du}{\cos(x)}\right)$

$= \int (1-u^2)u^4\,du$

$= \int (u^4-u^6)\,du$

$= \frac{u^5}{5}-\frac{u^7}{7}+C$

$$\boxed{= \frac{\sin^5(x)}{5}-\frac{\sin^7(x)}{7}+C}$$

check!?

Let $u=\sin(x)$

$\frac{du}{dx}=\cos(x)$

$du=\frac{du}{dx}\cdot dx$

$du=\cos(x)dx$

$\frac{du}{\cos(x)}=dx$

$\cos^2(x)+\sin^2(x)=1$

$\cos^2(x)=1-\sin^2(x)$

m is odd, use (1)

Ex.2 Integrate: $\int \frac{\sin^5(t)}{\sqrt{\cos(t)}}dt = \int [\cos(t)]^{-1/2} \cdot \sin^4(t) \cdot \sin(t)\,dt$

"odd guy out"

$= \int [\cos(t)]^{-1/2} \cdot [\sin^2(t)]^2 \cdot \sin(t)\,dt$

$= \int (u)^{-1/2} \cdot [1-u^2]^2 \cdot [\sin(t)] \cdot \left(\frac{du}{-\sin(t)}\right)$

$= -\int u^{-1/2} \cdot (1 - 2u^2 + u^4)\,du$

$= -\int (u^{-1/2} - 2u^{3/2} + u^{7/2})\,du$

$= -\left(\frac{2}{1}u^{1/2} - 2 \cdot \frac{2}{5}u^{5/2} + \frac{2}{9}u^{9/2}\right) + C$

$= -2u^{1/2} + \frac{4}{5}u^{5/2} - \frac{2}{9}u^{9/2} + C$

$\boxed{= -\frac{2}{9}\cos^{9/2}(t) + \frac{4}{5}\cos^{5/2}(t) - 2\cos^{1/2}(t) + C}$

Let
$u = \cos(t)$
$\frac{du}{dt} = -\sin(t)$
$du = \frac{du}{dt} \cdot dt$
$du = -\sin(t)dt$
$\frac{du}{-\sin(t)} = dt$

$\sin^2(x) + \cos^2(x) = 1$
$\sin^2(x) = 1 - \cos^2(x)$
$\sin^2(x) = 1 - u^2$

$(1-u^2)^2 = (1-u^2)(1-u^2)$
$= 1 - 2u^2 + u^4$

check:

m is even, n=0, use (3)

Ex.3 Integrate: $\int \sin^4(2\theta)d\theta = \int [\sin^2(2\theta)]\cdot[\sin^2(2\theta)]\,d\theta$

use "Power Reducing" identity

$= \int \left[\frac{1-\cos(4\theta)}{2}\right]\cdot\left[\frac{1-\cos(4\theta)}{2}\right]d\theta$

$= \frac{1}{4}\int [1-2\cos(4\theta)+\cos^2(4\theta)]\,d\theta$

$= \frac{1}{4}\int 1\,d\theta - \frac{1}{2}\int \cos(4\theta) + \frac{1}{4}\int \cos^2(4\theta)\,d\theta$

$= \frac{1}{4}\theta - \frac{1}{2}\int \cos(u)\cdot\left(\frac{du}{4}\right) + \frac{1}{4}\int \left[\frac{1+\cos(8\theta)}{2}\right]d\theta$

$= \frac{1}{4}\theta - \frac{1}{8}\int \cos(u)\,du + \frac{1}{8}\int [1+\cos(8\theta)]\,d\theta$

$= \frac{1}{4}\theta - \frac{1}{8}[\sin(u)] + \frac{1}{8}\int 1\,d\theta + \frac{1}{8}\int \cos(8\theta)\,d\theta$

$= \frac{1}{4}\theta - \frac{1}{8}\sin(4\theta) + \frac{1}{8}\theta + \frac{1}{8}\int \cos(z)\cdot\left(\frac{dz}{8}\right)$

$= \frac{2}{8}\theta + \frac{1}{8}\theta - \frac{1}{8}\sin(4\theta) + \frac{1}{64}\int \cos(z)\,dz$

$= \frac{3}{8}\theta - \frac{1}{8}\sin(4\theta) + \frac{1}{64}[\sin(z)] + C$

$$\boxed{= \frac{3}{8}\theta - \frac{1}{8}\sin(4\theta) + \frac{1}{64}\sin(8\theta) + C}$$

check: ??

use "Power Reducing"

$\sin^2(x) = \frac{1-\cos(2x)}{2}$

Let $u = 4\theta$

$\frac{du}{d\theta} = 4$

$\frac{du}{4} = d\theta$

Use "Power Reducing"

$\cos^2(x) = \frac{1+\cos(2x)}{2}$

Let $z = 8\theta$

$\frac{dz}{d\theta} = 8$

$\frac{dz}{8} = d\theta$

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<u>*Type II:*</u> $\int \tan^n(x)\sec^m(x)dx$

We can use $\sin^2(\theta)+\cos^2(\theta)=1$ to make two other Pythagorean trigonometric identities. These are $\sec^2(\theta)=1+\tan^2(\theta)$ & $\tan^2(\theta)=\sec^2(\theta)-1$.
In order to use these identities in combination with the substitution technique, we'll need to remember the follow derivatives:

$$\frac{d}{d\theta}[\tan(\theta)]=\sec^2(\theta) \quad \& \quad \frac{d}{d\theta}[\sec(\theta)]=\sec(\theta)\tan(\theta)$$

Guidelines for Evaluating Integrals Involving Secant and Tangent

1. If the power of the secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then expand and integrate.

$$\int \overset{\text{Even}}{\sec^{2k}} x \tan^n x\,dx = \int \overbrace{(\sec^2 x)^{k-1}}^{\text{Convert to tangents}} \tan^n x \overbrace{\sec^2 x\,dx}^{\text{Save for } du} = \int (1+\tan^2 x)^{k-1}\tan^n x \sec^2 x\,dx$$

2. If the power of the tangent is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then expand and integrate.

$$\int \sec^m x \overbrace{\tan^{2k+1}}^{\text{Odd}} x\,dx = \int \sec^{m-1} x \overbrace{(\tan^2 x)^k}^{\text{Convert to secants}} \overbrace{\sec x \tan x\,dx}^{\text{Save for } du} = \int \sec^{m-1} x(\sec^2 x - 1)^k \sec x \tan x\,dx$$

3. If there are no secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor, then expand and repeat if necessary.

$$\int \tan^n x\,dx = \int \tan^{n-2} x\overbrace{(\tan^2 x)}^{\text{Convert to secants}}\,dx = \int \tan^{n-2} x(\sec^2 x - 1)\,dx$$

4. If the integral is of the form $\int \sec^m x\,dx$, where m is odd and positive, use integration by parts, as illustrated in Example 5 in the preceding section.
5. If none of the first four guidelines applies, try converting to sines and cosines.

m is even, use (1)

Ex.4 Integrate: $\int \sec^4\left(\frac{x}{2}\right)\tan\left(\frac{x}{2}\right)dx = \int \sec^2\left(\frac{x}{2}\right)\cdot\tan\left(\frac{x}{2}\right)\cdot\sec^2\left(\frac{x}{2}\right)dx$

$= \int [1+\tan^2(\frac{x}{2})]\cdot\tan(\frac{x}{2})\cdot\sec^2(\frac{x}{2})dx$

$= \int [1+u^2]\cdot(u)\cdot\sec^2(\frac{x}{2})\cdot\left(\frac{2\,du}{\sec^2(\frac{x}{2})}\right)$

$= 2\int [u+u^3]\cdot du$

$= 2\left[\frac{u^2}{2}+\frac{u^4}{4}\right]+C$

$= u + \frac{u^4}{2} + C$

$\boxed{= \tan^2(\frac{x}{2}) + \frac{\tan^4(\frac{x}{2})}{2} + C}$

check!??

Use
$\sec^2(\theta)=1+\tan^2(\theta)$

Let $u=\tan(\frac{x}{2})$

$\frac{du}{dx}=[\sec^2(\frac{x}{2})]\cdot(\frac{1}{2})$

$du=\frac{du}{\cancel{dx}}\cdot\cancel{dx}$

$du=\frac{1}{2}\sec^2(\frac{x}{2})dx$

$\frac{2\,du}{\sec^2(\frac{x}{2})}=dx$

n is odd, use (2)

Ex.5 Integrate: $\int \tan^{3}\left(\frac{\pi x}{2}\right)\sec^{2}\left(\frac{\pi x}{2}\right)dx$

$$= \int \tan^2\left(\frac{\pi x}{2}\right)\cdot \sec\left(\frac{\pi x}{2}\right)\cdot \left[\sec\left(\frac{\pi x}{2}\right)\tan\left(\frac{\pi x}{2}\right)\right]dx$$

$$= \int \left[\sec^2\left(\frac{\pi x}{2}\right)-1\right]\cdot \sec\left(\frac{\pi x}{2}\right)\cdot \left[\sec\left(\frac{\pi x}{2}\right)\tan\left(\frac{\pi x}{2}\right)\right]dx$$

$$= \int [u^2-1]\cdot(u)\cdot\left[\sec\left(\frac{\pi x}{2}\right)\tan\left(\frac{\pi x}{2}\right)\right]\cdot\left[\frac{2\,du}{\pi \sec\left(\frac{\pi x}{2}\right)\tan\left(\frac{\pi x}{2}\right)}\right]$$

$$= \frac{2}{\pi}\int (u^3-u)\,du$$

$$= \frac{2}{\pi}\left(\frac{u^4}{4}-\frac{u^2}{2}\right)+C$$

$$= \frac{1}{\pi}\left(\frac{u^4}{2}-u^2\right)+C$$

$$\boxed{= \frac{1}{\pi}\left[\frac{\sec^4\left(\frac{\pi x}{2}\right)}{2}-\sec^2\left(\frac{\pi x}{2}\right)\right]+C}$$

<u>check</u>: ??

use

$\tan^2(\theta)=\sec^2(\theta)-1$

Let

$u=\sec\left(\frac{\pi x}{2}\right)$

$\frac{du}{dx}=\sec\left(\frac{\pi x}{2}\right)\tan\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}$

$du=\frac{du}{dx}\cdot dx$

$du=\sec\left(\frac{\pi x}{2}\right)\tan\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}\cdot dx$

$\frac{2\,du}{\pi \sec\left(\frac{\pi x}{2}\right)\tan\left(\frac{\pi x}{2}\right)}=dx$

n is even, m=0 → use (3)

Ex.6 Evaluate: $\int \tan^{6}(x)\,dx = \int [\tan^4(x)]\cdot[\tan^2(x)]\,dx$

$= \int \tan^4(x)\cdot[\sec^2(x)-1]\,dx$

$= \int \tan^4(x)\sec^2 x\,dx - \int \tan^4(x)\,dx$

$= \int (u^4)\cdot \sec^2(x)\cdot\left(\frac{du}{\sec^2(x)}\right) - \int [\tan^2(x)]\cdot[\tan^2(x)]\,dx$

$= \int u^4\,du - \int \tan^2(x)\cdot[\sec^2(x)-1]\,dx$

$= \frac{u^5}{5} - \int \tan^2(x)\sec^2(x)\,dx + \int \tan^2(x)\,dx$

$= \frac{\tan^5(x)}{5} - \int (u^2)\cdot\sec^2(x)\cdot\left(\frac{du}{\sec^2(x)}\right) + \int [\sec^2(x)-1]\,dx$

$= \frac{\tan^5(x)}{5} - \int u^2\,du + \int \sec^2(x)\,dx - \int 1\,dx$

$= \frac{\tan^5(x)}{5} - \frac{u^3}{3} + \tan(x) - x + C$

$\boxed{= \frac{\tan^5(x)}{5} - \frac{\tan^3(x)}{3} + \tan(x) - x + C}$

Check: ??

Wow!!

use

$\tan^2(\theta) = \sec^2(\theta) - 1$

Repeat if needed!

Helps Reduce Powers of tangent

Let $u = \tan(x)$

$\frac{du}{dx} = \sec^2(x)$

$\frac{du}{\sec^2(x)} = dx$

m is odd, n = 0, → use (4), "Parts"

Ex.7 Evaluate: $\int \sec^{3}(x)\,dx$

$\int u\,dv = uv - \int v\,du$

Let $u = \sec(x)$

$\frac{du}{dx} = \sec(x)\tan(x)$

$du = \frac{du}{dx} \cdot dx$

$du = \sec(x)\tan(x)\,dx$

Let $dv = \sec^2(x)\,dx$

$\int dv = \int \sec^2(x)\,dx$

$v = \tan(x) + c$

$v = \tan(x)$

$\tan^2(\theta) = \sec^2(\theta) - 1$

$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + c$

$= \int [\sec(x)] \cdot [\sec^2(x)\,dx] = \int u\,dv$

$= (\;\;u\;\;)(\;\;v\;\;) - \int (\;\;v\;\;)\cdot(\;\;du\;\;)$

$= [\sec(x)] \cdot [\tan(x)] - \int [\tan(x)] \cdot [\sec(x)\tan(x)\,dx]$

$= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx$

$= \sec(x)\tan(x) - \int \sec(x)[\sec^2(x) - 1]\,dx$

$= \sec(x)\tan(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx$

$= \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| - \int \sec^3(x)\,dx$ ★

"Loop," so add

$\int \sec^3(x)\,dx = \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| - \int \sec^3(x)\,dx$

$+ \int \sec^3(x)\,dx \quad = \quad + \int \sec^3(x)\,dx$

$2\int \sec^3(x)\,dx = \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| + C$

$\frac{1}{2} \cdot 2\int \sec^3(x)\,dx = \frac{1}{2}[\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|] + C$

$\boxed{\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x) + \tan(x)] + c}$

Check: ??

Zoinks!!

(5) convert to sines & cosines

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Ex.8 Integrate: $\int \frac{\tan^2(x)}{\sec^5(x)}dx = \int \frac{\left[\frac{\sin(x)}{\cos(x)}\right]^2}{\left[\frac{1}{\cos(x)}\right]^5}dx$

$= \int \frac{\sin^2(x)}{\cos^2(x)} \cdot \frac{\cos^5(x)}{1}dx$

n is odd, use (2)

$= \int \sin^2(x)\cos^3(x)dx$

$= \int [\sin^2(x)]\cdot[\cos^2(x)]\cdot\cos(x)dx$

$= \int [\sin^2(x)]\cdot[1-\sin^2(x)]\cdot\cos(x)dx$

$= \int [u^2]\cdot[1-u^2]\cdot\cos(x)\cdot\left(\frac{du}{\cos(x)}\right)$

$= \int (u^2-u^4)du$

$= \frac{u^3}{3} - \frac{u^5}{5} + C$

$\boxed{= \frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C}$

check: ??

use

$\cos^2(\theta) = 1-\sin^2(\theta)$

Let $u = \sin(x)$

$\frac{du}{dx} = \cos(x)$

$\frac{du}{\cos(x)} = dx$

Are you comfortable manipulating your favorite Pythagorean trig identity?

$\sin^2\theta + \cos^2\theta = 1$

$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$

$1+\cot^2\theta = \csc^2\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$

$\tan^2\theta + 1 = \sec^2\theta$

Type III: *Integrals Involving Sine-Cosine Products with Different Angles*

In concert with the substitution technique, we'll use the following three identities:

(c) $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$

(a) $\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$

(b) $\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$

← use (c)

Ex.9 Integrate: $\int \sin(-4x)\cos(3x)\,dx$

$= \int \frac{1}{2}\left[\sin(-7x)+\sin(-x)\right]dx$

$= \frac{1}{2}\cdot\int\sin(-7x)\,dx + \frac{1}{2}\cdot\int\sin(-x)\,dx$ **

$= -\frac{1}{2}\int\sin(7x)\,dx - \frac{1}{2}\int\sin(x)\,dx$

$= -\frac{1}{2}\int\sin(u)\cdot\left(\frac{du}{7}\right) - \frac{1}{2}\cdot\left[-\cos(x)\right] + C$

$= -\frac{1}{14}\int\sin(u)\,du + \frac{1}{2}\cos(x) + C$

$= -\frac{1}{14}\cdot\left[-\cos(u)\right] + \frac{1}{2}\cos(x) + C$

$$\boxed{= \frac{\cos(7x)}{14} + \frac{\cos(x)}{2} + C}$$

check: ??

use

$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$

Let

$\alpha = -4x$ & $\beta = 3x$

$\alpha - \beta = -4x - 3x$

$\alpha - \beta = -7x$

$\alpha + \beta = -4x + 3x$

$\alpha + \beta = -x$

** sine is an "odd" function,

$\sin(-x) = -\sin(x)$

cosine is an "even" function,

$\cos(-x) = \cos(x)$

Let

$u = 7x$

$\frac{du}{dx} = 7$

$\frac{du}{7} = dx$